it turns out to be sufficient that all the roots $\lambda_1^{(1)},\ldots,\lambda_1^{(s)}$ of the s th-degree equation

$$\left|\frac{\partial P_r}{\partial h_q} - \lambda_1 T \delta_{qr}\right| = 0 \qquad (4.14)$$

have negative real parts. It can be shown that a similar correspondence is preserved also in the autonomous case as well as in the more complex cases when the critical characteristic indices for $\mu = 0$ have nonprime elementary divisors.

BIBLIOGRAPHY

1. Neimark, Iu. I. and Shil'nikov, L. P., On the application of the small parameter method to systems of differential equations with discontinuous right-hand sides. Izv. Akad. Nauk SSSR, OTN, Mekhanika i Mashinostroenie, №6, 1959.
2. Malkin, I. G., Certain Problems in the Theory of Nonlinear Oscillations. Moscow, Gostekhizdat, 1956.
3. Neimark, Iu. I., The method of point transformations in the theory of nonlinear oscillations, I, II, III. Izv. Vuzov, Radiofizika, №№1, 2, 5, 6, 1958.
4. Kolovskii, M. Z., On conditions for the existence of periodic solutions of systems of differential equations with discontinuous right-hand sides containing a small parameter. PMM Vol. 24, №4, 1960.
5. Mikusiński, J. G. and Sikorski, R., The elementary theory of distributions, I, II. Rozprawy Mat., Vol. 12, 1957, 54 pp.; Vol. 25, 1961, 47 pp.
6. Rozenvasser, E. N., Oscillations in Nonlinear Systems. Moscow, "Nauka", 1969.

Translated by N.H.C.

UDC 532.593

ON STEADY CAPILLARY-GRAVITATIONAL WAVES OF FINITE AMPLITUDE AT THE SURFACE OF FLUID OVER AN UNDULATING BED

PMM Vol. 36, №6, 1972, pp. 1070-1085
Ia. I. SEKERZH-ZEN'KOVICH
(Moscow)
(Received July 26, 1972)

The problem of steady capillary-gravitational plane waves of finite amplitude at the surface of a stream of perfect incompressible fluid over an undulating bed under constant surface pressure is considered. The intersection of the undulating bed surface with the vertical plane is assumed to be a periodic curve which is called the bed line and specified by an infinite trigonometric series. An exact solution of this problem, which reduces it to a system of nonlinear integral and transcedental equations, is presented. The theorem of existence and uniqueness of solution of that system is obtained on the assumption of smallness of the bed line amplitude. The method of proving this theorem is indicated and the method of deriving solutions with any degree of approxima-

tion is described. The solution is constructed in the form of series expansions in terms of a small dimensionless parameter proportional to the amplitude of the bed line first harmonic. First three approximations are worked out right to the end. The approximate equation of wave profile is presented.

The particular case of a wave whose arc length is equal to that of a steady free linear wave, which corresponds to the specified stream velocity over a horizontal flat bed for constant pressure is also considered. In this case the parameter of the basic integral equation is equal to one of the eigenvalues of the kernel of this equation, and the solution is constructed in the form of series expansions in powers of the cubic root of the small parameter mentioned above.

A similar problem was first analyzed in [1] by the Levi-Civita method by which it was reduced to solving nonlinear differential equations, without, however, considering the particular case mentioned above. Capillary-gravitational waves over an undulating bed were considered in [2, 3]. Only the proof of the theorem of existence of solution by methods of functional analysis for high stream velocities is given in [2]. The topolgical proof of existence and uniqueness of solution and the algorithm of its derivation is given in [3] but the calculation of approximations is only outlined.

In the present paper the equation of the bed line, unlike in [3], is of a form which makes it possible to represent any approximations as finite sums, and the basic system of nonlinear integral and transcedental equations is analyzed by the analytical method of Liapunov-Schmidt and its developments.

The waves considered here and in referenced works are induced by the undulating bed surface and, if the bed is flat, they cease to exist and the stream becomes uniform. We shall call such waves induced, as distinct from those which exist in the case of a flat bed at particular stream velocities.

1. Statement of the problem and derivation of basic equations.

Let us consider a stable plane-parallel motion of a perfect incompressible heavy fluid bounded from above by a free surface under pressure p which is assumed constant and equal p_0. From below the fluid is bounded by an undulating bed, whose intersection with the vertical flow plane is specified by a certain periodic and twice-differentiable curve L which is called the bed line. Line L is assumed to be symmetric about vertical lines drawn through its crests and troughs. Let us assume that the stream flows at a specified mean horizontal velocity c at $y = 0$ (see below) from left to right. Owing to the periodicity of the bed line, the fluid free surface assumes the form of a stationary periodic wave, which in coordinates attached to the progressing wave moves at velocity $-c$.

Let the crests of the unknown wave and of curve L lie on the same verical line and the two be symmetric about that line, and line L be, also, symmetric about the vertical line passing through the middle of its trough. We superpose the y-axis of an orthogonal system of coordinates xy on the axis of wave symmetry and direct it upward. We locate the coordinate origin O at the intersection point of the y-axis with line L and direct the x-axis from left to right along the tangent to the bed line L. Let the period (or the wave length) of line L along x be λ. Along the distance between two adjacent wave crests there is at least one trough (in the general case, several crests and troughs may exist in a single wave length). It is assumed that line L has horizontal tangents at points $x = 0$ and $x = \pm {}^1/_2\, \lambda$. The angle of a tangent of line L to the x-axis is specified in the form of function $\Theta(s)$ of arc lengths measured from zero. The direction of increasing arc length s is taken as the positive direction along the tangent. We denote by $2l$ the arc length of line L in a single period along x, i.e. for $0 \leqslant x \leqslant \lambda$. For $x = -\frac{1}{2}\lambda$ and $x = \frac{1}{2}\lambda$ the arc lengths are, respectively, $s = -l$ and $s = l$. Since $\Theta(s)$ is a continuous function of s which changes its sign at transition through crest tips and the middle of the troughs, then

$$\Theta(0) = \Theta(l) = \Theta(-l) = 0 \tag{1.1}$$

By virtue of the imposed symmetry condition, we have

$$\Theta(-l + s) = -\Theta(l - s) \tag{1.2}$$

Actual solution of this problem requires the availability of analytical expression for functions $\Theta(s)$. Assuming that the slope of line L is small, we consider in accordance with the condition of periodicity and conditions (1.1) and (1.2) that the function $\Theta(s)$ is defined by the trigonometric series

$$\Theta(s) = \sum_{n=1}^{\infty} \varepsilon^n \beta_n \sin \frac{n\pi s}{l} \tag{1.3}$$

where ε is a small positive dimensionless parameter, β_n are specified real numbers, and the series

$$\sum_{n=1}^{\infty} \varepsilon^n \beta_n$$

is convergent in a circle of radius $\varepsilon_0 > 0$. Having determined function $\Theta(s)$, we can obtain the parametric equation of the bed line in the form

$$x = \int_0^s \cos\Theta(s)\, ds, \qquad y = \int_0^s \sin\Theta(s)\, ds \tag{1.4}$$

In this case the wave length λ of line L is, obviously, determined by formula

$$\lambda = \int_0^{2l} \cos\Theta(s)\, ds \tag{1.5}$$

It follows from formulas (1.3) and (1.5) that λ is defined by the known function of ε:

$$\lambda = \lambda_0 + \sum_{n=1}^{\infty} \lambda_n \varepsilon^n, \qquad \lambda_0 = 2l, \qquad \lambda_1 = 0, \qquad \lambda_2 = -\frac{\beta_1^2 l}{2}, \qquad \lambda_3 = 0 \tag{1.6}$$

where λ_n $(n = 4, 5, \ldots)$ are polynomials of β_i. The length of the unknown steady wave over the undulating bed is assumed to be also equal λ.

We take the flow plane xy as the plane of the complex variable $z = x + iy$. Let φ be the velocity potential, ψ the stream function, $w = \varphi + i\psi$ the complex velocity potential, and U and V the projections of the velocity vector q on the axes of coordinates. We then have

$$\frac{dw}{dz} = -U + iV, \qquad U = -\frac{\partial\varphi}{\partial x}, \qquad V = -\frac{\partial\varphi}{\partial y}$$

For the derivation of boundary conditions for the basic equations of this problem we, first of all, conformally map the region of a single wave bounded by two vertical lines and two wave-like curves from above and below, onto the rectangle

$$0 \leqslant \varphi \leqslant \varphi_0, \qquad 0 \leqslant \psi \leqslant \psi_0$$

in the plane w (here $\psi = \psi_0$ is the stream rate of flow per unit of time, $\varphi = 0$ and $\varphi = \varphi_0$ at $x = 0$ and $x = \lambda$), respectively), and then map this rectangle inside the circular ring whose center lies at the zero of plane $u = u_1 + iu_2$. The last transformation is defined by formula

$$w = \frac{\varphi_0}{2\pi i} \ln u \tag{1.7}$$

The segment $0 \leqslant \varphi \leqslant \varphi_0$ of the free surface is now represented by the external circle of unit radius and that of the bed line by the inner circle of radius $r_0 = \exp(-2\pi\psi_0 / \varphi_0)$ smaller than unity. The ring is slit along segment $(r_0, 1)$. For the derivation of solution we assume that ψ_0 / φ_0 and, consequently, also r_0 are specified and independent of ε (see (1.3)). The mapping of this ring from plane u onto the region of a single wave in the z-plane is defined by the relationship

$$\frac{dz}{du} = -\frac{\lambda}{2\pi i} \frac{f(u)}{u} \tag{1.8}$$

Function $f(u)$, which is holomorphic, is represented within the considered ring in plane u by a Laurent expansion. Owing to the symmetry of the wave and bed line, the coefficients of this expansion must be real. Introducing, as usually [1], function

$$\omega(u) = \Phi + i\tau = -i \ln f(u) \tag{1.9}$$

by virtue of (1.7) and (1.8) and setting $\varphi_0 = c\lambda$, we obtain

$$dw/dz = -ce^{\tau - i\Phi} \tag{1.10}$$

It follows from this that throughout the stream function Φ is equal to the angle of inclination of the velocity vector $\mathbf{q}$ to the x-axis, and that

$$q = |\mathbf{q}| = ce^{\tau} \tag{1.11}$$

For $u = e^{i\theta}$ (θ is the angle of inclination of the radius vector to the u_1-axis) from (1.9) and (1.8) we obtain a differential relationship in which we separate the real and imaginary parts and, after Integration, obtain for the wave profile the parameteric equation

$$x = -\frac{\lambda}{2\pi} \int_0^\theta e^{-\tau(\eta)} \cos \Phi(\eta)\, d\eta, \qquad y = -\frac{\lambda}{2\pi} \int_0^\theta e^{-\tau(\eta)} \sin \Phi(\eta)\, d\eta \tag{1.12}$$

For the determination of y we transfer the coordinate origin to the wave crest tip, and in (1.12) set $\tau(\eta) = \tau(1, \eta)$ and $\Phi(\eta) = \Phi(1, \eta)$.

Formulas (1.12) imply that for solving the problem we must determine not only $\Phi(\theta)$ but, also, $\tau(\theta)$. Owing to the symmetry of the unkown wave about the vertical line passing through its crest, function $\tau(\theta)$ is even and function $\Phi(\theta)$ is odd. Hence they can be represented by the following trigonometric series:

$$-\tau(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta, \qquad \Phi(\theta) = \sum_{n=1}^{\infty} B_n \sin n\theta \tag{1.13}$$

It is known from the theory of analytic functions that at the external circumference the following relationships:

$$-\tau(\theta) - A_0 = \int_0^{2\pi} K(\eta, \theta) \frac{d\Phi}{d\eta} d\eta - 2\int_0^{2\pi} N(\eta, \theta) \frac{d\Phi^*}{d\eta} d\eta$$

$$K(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{\nu_n'}, \qquad N(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{\nu_n^*}$$

$$\nu_n' = n \frac{r_0^{-n} - r_0^n}{r_0^{-n} + r_0^n}, \qquad \nu_n^* = n(r_0^{-n} - r_0^n), \qquad \frac{4}{\nu_n^{*2}} = \frac{1}{\nu_n'^2} - \frac{1}{n^2}$$

$$\Phi(\theta) = \int_0^{2\pi} K_0(\eta, \theta) \frac{d\tau}{d\eta} d\eta + 2\int_0^{2\pi} M(\eta, \theta) \frac{d\Phi^*}{d\eta} d\eta \tag{1.14}$$

$$K_0(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\eta \sin n\theta}{\nu_n''}, \qquad M(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\eta \sin n\theta}{\nu_n^*}$$

$$\nu_n'' = n \frac{r_0^{-n} + r_0^n}{r_0^{-n} - r_0^n}, \qquad \nu_n^{**} = n(r_0^n + r_0^{-n}), \qquad \nu_n' \nu_n'' = n^2$$

based on Will's formulas for a ring and the generalizing relationships of Dini are valid. In the relationships (1.14)

$$\tau^*(\theta) = \tau(r_0, \theta), \qquad \Phi^*(\theta) = \Phi(r_0, \theta)$$

Owing to the symmetry of the bed line formulas (1.13) are valid for these functions but with different coefficients A_n and B_n $(n = 1, 2, 3, \ldots)$. Passing to the boundary condition at the surface, we take for the latter the Bernoulli integral

$$p / \rho = C - gy - {}^1/_2 q^2 \tag{1.15}$$

where C is a constant, g is the acceleration of gravity, and ρ is the density. Pressure differences at the free surface are balanced by the vertical components of surface tension. For these forces by the Laplace law we have

$$p - p_0 = \pm \mu / R \tag{1.16}$$

where p is the pressure from the inside of fluid, $p_0 = \text{const}$ is the pressure from outside of the fluid, μ is the capillarity constant, and R is the radius of curvature at points of the free surface. Expressing the curvature in terms of $d\Phi / d\theta$, we obtain

$$p - p_0 = \frac{2\pi\mu}{\lambda c} q \frac{d\Phi}{d\theta} \tag{1.17}$$

Substituting the expression for p from (1.17) into (1.15), we have

$$\frac{d\Phi}{d\theta} = \nu \left[\delta e^{-\tau} - e^{\tau} - \frac{2\pi}{\lambda} \varkappa y e^{-\tau}\right] \tag{1.18}$$

$$\nu = \frac{\lambda c^2 \rho}{4\pi\mu}, \qquad \delta = \frac{2(C\rho - p_0)}{\rho c^2}, \qquad \varkappa = \frac{g\lambda}{\pi c^2} \tag{1.19}$$

where y is determined by the second formula of (1.12). Separating in the right-hand side of (1.18) the terms which are linear with respect to Φ and τ, we obtain

$$\frac{d\Phi}{d\theta} = \nu \left\{\delta - 1 - (\delta + 1)\tau + \varkappa \int_0^{\theta} \Phi(\eta)\, d\eta + F[\tau, \Phi, \delta]\right\} \tag{1.20}$$

$$F[\tau, \Phi, \delta] = \delta(e^{-\tau} - 1 + \tau) - (e^{\tau} - 1 - \tau) + \varkappa e^{-\tau}\int_0^\theta [e^{-\tau(\eta)} \sin\Phi(\eta) - \Phi(\eta)]\, d\eta + \varkappa e^{-\tau}\int_0^\theta \Phi(\eta)\, d\eta - \varkappa\int_0^\theta \Phi(\eta)\, d\eta \quad (1.21)$$

Let us determine the parameters in Eq. (1.20) more accurately. It follows from (1.19) and (1.6) that

$$\nu = \nu^{(0)} + \sum_{n=1}^{\infty} \nu^{(n)}\varepsilon^n, \quad \nu^{(0)} = \frac{c^2\rho\lambda_0}{4\pi\mu}, \quad \nu^{(n)} = \frac{\nu^{(0)}}{\lambda_0}\lambda_n$$

$$\varkappa = \varkappa_0 + \sum_{n=1}^{\infty} \varkappa_n\varepsilon^n, \quad \varkappa_0 = \frac{g\lambda_0}{\pi c^2}, \quad \varkappa_n = \frac{\varkappa_0}{\lambda_0}\lambda_n \quad (1.22)$$

By virtue of (1.22) Eq. (1.20) assumes the form

$$\frac{d\Phi}{d\theta} = \nu^{(0)}\Big\{\delta - 1 - (\delta + 1)\tau + \varkappa_0\int_0^\theta \Phi(\eta)\, d\eta + \sum_{n=1}^{\infty} \varkappa_n\varepsilon^n\int_0^\theta \Phi(\eta)\, d\eta + F[\tau, \Phi, \delta]\Big\} + \sum_{n=1}^{\infty} \nu^{(n)}\varepsilon^n\{\ \} \quad (1.23)$$

where the expression omitted in the second set of braces is the same as that appearing in the first. We transform the terms linear with respect to functions and parameter ε in braces in (1.23) by using formulas (1.14) and integration by parts. We then combine in the first braces the terms with the same integrand functions $d\Phi / d\eta$ and different kernels $K(\eta, \theta)$ and $K_2(\eta, \theta)$ from (1.26). Since the velocity c is specified, the parameters $\nu^{(0)}$ and $\varkappa_0$ are fixed, and δ is determined by the periodicity condition $\Phi(\theta + 2\pi) = \Phi(\theta)$. Since the right-hand side of Eq. (1.23) contains parameter ε, hence the solution and, consequently, also δ depend on ε. We set

$$\delta = \delta_0 + \delta'(\varepsilon) \quad (1.24)$$

From the periodicity condition at $\varepsilon \to 0$ we find that $\delta_0 = 1$, since $\delta'(\varepsilon)$, as well as the solution tend to vanish. After all these transformations with allowance for (1.24), Eq. (1.23) assumes the final form

$$\zeta(\theta) = \nu^{(0)}\Big\{\int_0^{2\pi} K^*(\eta, \theta)\zeta(\eta)\, d\eta + \delta'(\varepsilon) - 2(2 + \delta'(\varepsilon))\int_0^{2\pi} N(\eta, \theta)\zeta^*(\eta)\, d\eta + (2 + \delta'(\varepsilon))A_0(\varepsilon) + \delta'(\varepsilon)\int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\, d\eta + \varkappa_0\int_0^{2\pi} K_2(\eta, 0)\zeta(\eta)\, d\eta + \sum_{n=1}^{\infty} \varkappa_n\varepsilon^n\int_0^\theta \Phi(\eta)\, d\eta + F[\tau, \Phi, 1 + \delta'(\varepsilon)]\Big\} +$$

$$\sum_{n=1}^{\infty} \nu^{(n)}\varepsilon^n \left\{2\int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\,d\eta - \varkappa_0 \int_0^{2\pi} K_2(\eta, \theta)\zeta(\eta)\,d\eta + \ldots\right\} \qquad (1.25)$$

(the dots in the second set of braces stand for the last seven terms of the same form as those appearing in the first set).

In the last equation

$$\zeta(\theta) = d\Phi / d\theta, \quad \zeta^*(\theta) = d\Phi^* / d\theta$$

$$K_2(\eta, \theta) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{n^2}, \qquad K^*(\eta, \theta) = \sum_{n=1}^{\infty} \frac{\varphi_n(\eta)\,\varphi_n(\theta)}{\nu_n}$$

$$\nu_n = \frac{n^2}{2\nu_n'' - \varkappa_0}, \qquad \varphi_n(\theta) = \frac{\cos n\theta}{\sqrt{\pi}} \qquad (1.26)$$

where ν_n are eigenvalues and $\varphi_n(\theta)$ are eigenfunctions of kernel $K^*(\eta, \theta)$. The condition of periodicity of function $\Phi(\theta)$ yields the relationship

$$\delta'(\varepsilon) = -\varkappa_0 \int_0^{2\pi} K_2(\eta, 0)\zeta(\eta)\,d\eta - (2 + \delta'(\varepsilon))A_0 - \frac{1}{2\pi}\int_0^{2\pi} \Psi(\theta, \varepsilon)\,d\theta -$$

$$\frac{1}{2\pi\nu^{(0)}} \sum_{n=1}^{\infty} \nu^{(n)}\varepsilon^n \left\{\left[\delta'(\varepsilon) + (2 + \delta'(\varepsilon))A_0 + \varkappa_0 \int_0^{2\pi} K_2(\eta, 0)\zeta(\eta)\,d\eta\right] 2\pi + \int_0^{2\pi} \Psi(\theta, \varepsilon)\,d\theta\right\} \qquad (1.27)$$

where

$$\Psi(\theta, \varepsilon) = \sum_{n=1}^{\infty} \varkappa_n \varepsilon^n \int_0^{\theta} \Phi(\eta)\,d\eta + F[\tau, \Phi, 1 + \delta'(\varepsilon)] \qquad (1.28)$$

Let us turn to the boundary condition at the undulating bed for $r = r_0$. Obviously, the condition of flow along the bed contour must be satisfied. In the used here notation and by virtue of formula (1.3) this condition is of the form

$$\Phi^*(\theta) = \Theta[s(\theta)] = \sum_{n=1}^{\infty} \varepsilon^n \beta_n \sin\frac{n\pi s(\theta)}{l} \qquad (1.29)$$

To find the final form of this boundary condition it is necessary to determine for the conformal transformation considered here the dependence of the arc length s along the bed line on angle θ in the ring plane, i.e. to find function $s(\theta)$. We recall that for $r = r_0$

$$dz = -\frac{\lambda}{2\pi} e^{-\tau^*(\theta) + i\Phi^*(\theta)}\,d\theta$$

hence

$$ds = |dz| = -\frac{\lambda}{2\pi} e^{-\tau^*(\theta)}\,d\theta \qquad (1.30)$$

The minus sign in formula (1.30) is there in order to have positive increments of arc s to correspond to negative increments of angle θ. From (1.30) we have

$$s(\theta) = -\frac{\lambda}{2\pi}\int_0^{\theta} e^{-\tau^*(\eta)}\,d\eta \qquad (1.31)$$

We select coefficient $A_0(\varepsilon)$ so as to satisfy the condition specifying the length of the bed line to be equal $2l$ over one period. According to (1.31) this condition is defined by formula

$$2l = -\frac{\lambda}{2\pi}\int_0^{-2\pi} e^{-\tau^*(\eta)}d\eta$$

or

$$2le^{-A_0(\varepsilon)} = -\frac{\lambda}{2\pi}\int_0^{-2\pi} e^{-\tau^*(\eta)-A_0(\varepsilon)}d\eta$$

or

$$2le^{-A_0(\varepsilon)} = \frac{\lambda}{2\pi}\int_0^{2\pi} e^{-\tau^*(-\eta)-A_0(\varepsilon)}d\eta \tag{1.32}$$

and, by virtue of (1.13) for τ^* and Φ^*, $-\tau^*(-\eta)-A_0(\varepsilon)$ does not contain $A_0(\varepsilon)$. It follows from this that the expansion of $s(\theta)$ in powers of ε

$$s(\theta) = s_0(\theta) + \sum_{n=1}^{\infty} \varepsilon^n s_n(\theta) \tag{1.33}$$

contains only one secular term

$$s_0(\theta) = -l\pi^{-1}\theta \tag{1.34}$$

and, consequently,

$$s(\theta) = -\frac{l}{\pi}\theta + \sum_{n=1}^{\infty} \varepsilon^n s_n(\theta) = -\frac{l}{\pi}\theta + s'(\theta) \tag{1.35}$$

Differentiating (1.29) with respect to θ and taking into account (1.35), we obtain the boundary condition at the bed in the final form

$$\begin{gathered}\zeta^*(\theta) = -\varepsilon\beta_1\cos\theta + \\ \varepsilon\beta_1\left\{-\left[\cos\theta\left(\cos\frac{\pi}{l}s'(\theta)-1\right)+\sin\theta\sin\frac{\pi}{l}s'(\theta)\right]+\right. \\ \left.\frac{\pi}{l}\left[\cos\theta\cos\left(\frac{\pi}{l}s'(\theta)\right)+\sin\theta\sin\left(\frac{\pi}{l}s'(\theta)\right)\right]\frac{ds'(\theta)}{d\theta}\right\}+ \\ +\sum_{n=2}^{\infty}\varepsilon^n\beta_n\frac{n\pi}{l}\cos\frac{n\pi s(\theta)}{l}\frac{ds(\theta)}{d\theta}\end{gathered} \tag{1.36}$$

Function $\tau^*(\theta)$ in Eq. (1.31) is determined by formula

$$-\tau^*(\theta) - A_0 = -\int_0^{2\pi} K(\eta,\theta)\frac{d\Phi^*}{d\eta}d\eta + 2\int_0^{2\pi} N(\eta,\theta)\frac{d\Phi}{d\eta}d\eta \tag{1.37}$$

which is derived in a manner similar to that of the first of formulas (1.14).

The problem is thus reduced to the determination of functions

$$\zeta(\theta,\varepsilon) = \frac{d\Phi}{d\theta}, \quad \zeta^*(\theta,\varepsilon) = \frac{d\Phi^*}{d\theta}, \quad s(\theta,\varepsilon).$$

and of constants $\delta = 1 + \delta'(\varepsilon)$ and $A_0(\varepsilon)$ in the system of Eqs. (1.25), (1.27), (1.31), (1.32) and (1.36). Functions $\tau(\theta,\varepsilon)$ and $\tau^*(\theta,\varepsilon)$ are found from (1.14) and (1.37), and

$$\Phi(\theta,\varepsilon) = \int_0^{\theta}\zeta(\eta,\varepsilon)\,d\eta, \quad \Phi^*(\theta,\varepsilon) = \int_0^{\theta}\zeta^*(\eta,\varepsilon)\,d\eta \tag{1.38}$$

If in this system $\tau(\theta)$ and $\tau^*(\theta)$ are eliminated with the use of formulas (1.14) and

(1.37), and $\Phi(\theta, \varepsilon)$ and $\Phi^*(\theta, \varepsilon)$ are expressed in the form (1.38), then Eqs. (1.25), (1.31) and (1.36) become nonlinear integral equations in terms of $\zeta(\theta, \varepsilon)$, $\zeta^*(\theta, \varepsilon)$ and $s(\theta, \varepsilon)$, while Eqs. (1.27) and (1.32) become transcedental with respect to $\delta'(\varepsilon)$ and $A_0(\varepsilon)$ with functionals relative to the unknown functions. If, however, the sequence of determination of approximate solutions is taken into account, it is reasonable to consider only Eq. (1.25) as being a nonlinear differential equation with respect to $\zeta(\theta, \varepsilon)$; the remaining ones may be considered as being nonlinear transcedental equations with respect to functions $\zeta^*(\theta, \varepsilon)$ and $s(\theta, \varepsilon)$ and constants $\delta'(\varepsilon)$ and $A_0(\varepsilon)$ with linear operators and functionals relative to the unknown functions.

Two kinds of solutions must be considered: the first in which $\nu^{(0)} \neq \nu_n$, and the second when $\nu^{(0)} = \nu_n$. In the first case the solution is derived in the form of series expansions in powers of parameter ε, and in the second it is obtained in terms of powers of $\varepsilon^{1/3}$. In both of these cases for the coefficients of expansion of $\zeta(\theta, \varepsilon)$ we obtain Fredholm's linear integral equations of the second kind with kernel $K^*(\eta, \theta)$ and parameter $\nu^{(0)}$ A solvable system of linear algebraic equations is always obtained for the determination of coefficients of expansions of remaining unknown quantities. Equations defining the first coefficients of these expansions are analyzed in Sect. 2 for $\nu^{(0)} = \nu_n$.

Let us define the limit values of the unknown quantities for $\varepsilon \to 0$. The boundary conditions and equations of this problem will be satisfied for $\varepsilon \to 0$, if we set

$$\tau(r, \theta) \equiv 0, \quad \Phi(r, \theta) \equiv 0, \quad \delta = 1, \; A_0 = 0$$

From (1.6) and (1.33) follows that

$$\lim \lambda = \lambda_0 = 2l, \qquad \lim s(\theta) = -\frac{l}{\pi}\theta \quad \text{for} \quad \varepsilon \to 0$$

Mechanically this limit solution means that a uniform stream with horizontal free surface flows over a horizontal bed. The length of the region into which the ring interior is transformed in the z-plane is $\lambda_0 = 2l$, and the velocity is, by virtue of (1.11), equal c.

2. Solution of the linear problem. 2.1. Solution of the linear problem for $\nu^{(0)} = \nu_n$ and analysis of the kernel of integral equation (1.25). Expressing the solutions of Eqs. (1.25) and (1.27) in the form of series expansions in terms of powers of $\varepsilon^{1/3}$, we obtain

$$\zeta_1(\theta) = \nu^{(0)}\Big[\int_0^{2\pi} K^*(\eta, \theta)\zeta_1(\eta)\,d\eta + \delta_1 + 2A_{01} + \varkappa_0 \int_0^{2\pi} K_2(\eta, 0)\zeta_1(\eta)\,d\eta\Big] \quad (2.1)$$

$$\delta_1 = -\varkappa_0 \int_0^{2\pi} K_2(\eta, 0)\zeta_1(\eta)\,d\eta - 2A_{01} \quad (2.2)$$

The same system is obtained, if in (1.25) and (1.27) we set $\zeta^*(\theta) \equiv 0$, as in the case of a free wave over a flat bed, and linit these to linear terms only. Eliminating δ_1 from (2.2) and (2.1) and omitting the subscript, we obtain

$$\zeta(\theta) = \nu^{(0)} \int_0^{2\pi} K^*(\eta, \theta)\zeta(\eta)\,d\eta \quad (2.3)$$

This equation is a homogeneous linear Fredholm's equation of the second kind, hence, by the second Fredholm's theorem it has a nonzero solution for $\nu^{(0)} = \nu_n$, where ν_n is the eigenvalue of kernel $K^*(\eta, \theta)$. On the other hand, by virtue of (1.22) the parameter $\nu^{(0)} > 0$, while ν_n, according to (1.26), depends on n and $\varkappa_0$. Parameter $\varkappa_0$ is taken as fixed, hence it is necessary to investigate the dependence of ν_n on n for fixed $\varkappa_0$. A detailed analysis of this dependence appears in [5]. Here we present only its results.

Let us assume that n is fixed and investigate the relationship between $\nu^{(0)}$ and $\varkappa_0$ for which Eq. (2.3) has a nonzero solution. Setting $\nu^{(0)} = \nu_n$, from (1.26) we have

$$\frac{1}{\nu^{(0)}} = \frac{1}{n^2}(2\nu_n'' - \varkappa_0) \tag{2.4}$$

Substituting in this equation the expressions for $\nu^{(0)}$ and $\varkappa_0$ from (1.22), we obtain the known dependence between c^2 and λ_0

$$c^2 = \left(\frac{2\pi\mu n}{\lambda_0\rho} + \frac{g\lambda_0}{2\pi n}\right)\operatorname{th}\left(2\pi n\frac{h}{\lambda_0}\right) \tag{2.5}$$

Relationships (2.4) and (2.5) have also been analyzed in [5]. We present here only the results of investigation of solution of the linear problem in the form of the following theorems.

T h e o r e m 2.1. Let

$$\frac{1}{\nu^{(0)}} = \frac{1}{n^2}(2\nu_n'' - \varkappa_0)$$

where n is a fixed positive integrer. Then for all $\varkappa_0$ in the interval $0 < \varkappa_0 < 2\nu_n''$ Eq. (2.3) has the unique nontrivial solution

$$\zeta(\theta) = C_1\varphi_n(\theta) = \frac{C_1}{\sqrt{\pi}}\cos n\theta$$

If

$$\varkappa_0 = \varkappa_0^{(m)} = \frac{2(m^2\nu_n'' - n^2\nu_m'')}{m^2 - n^2}$$

(m is a positive integer), then

$$\zeta(\theta) = C_2\varphi_m(\theta) = \frac{C_2}{\sqrt{\pi}}\cos m\theta$$

is a particular nontrivial solution which is linearly independent of $\varphi_n(\theta)$, and

$$\zeta(\theta) = C_1\varphi_n(\theta) + C_2\varphi_m(\theta) = \frac{C_1}{\sqrt{\pi}}\cos n\theta + \frac{C_2}{\sqrt{\pi}}\cos m\theta$$

is the general solution.

We call $\varkappa_0 = \varkappa_0^{(m)}$ bifurcational parameters, and the waves, corresponding to these and determined by the solution in the form of a sum of two harmonics, are called double-waves. The related eigenvalue $\nu_n = \nu_m$ is for $\varkappa_0 = \varkappa_0^{(m)}$ double-valued.

T h e o r e m 2.2. The curve $c^2 = c^2(\lambda_0)$ which represents Eq. (2.5) has a vertical asymptote $\lambda_0 = 0$ and a horizontal one $c^2 = gh$. The value $c^2_{\min}$ which corresponds to $\lambda_0 = \lambda_0^*$, where λ_0^* is the positive root of some transcedental equation, lies in the first quadrant. The related value $\varkappa_0 = \varkappa_0^*$ is called critical. From (1.22) we have

$$\varkappa_0^* = \frac{g\lambda_0^*}{\pi c^2_{\min}}$$

The branch of curve $c^2 = c^2(\lambda_0)$ which corresponds to $0 < \lambda_0 < \lambda_0^*$ or to $0 < \varkappa_0 < \varkappa_0^*$ relates to waves called capillary-gravitational. Waves which occur for $\lambda_0 > \lambda_0^*$ or

$\varkappa_0^* < \varkappa_0 < 2\nu_n''$ are referred to as gravitational-capillary waves.

Theorem 2.3. For $\varkappa_0 = 0$ Eq. (2.3) assumes the form

$$\zeta(\theta) = 2\nu^{(0)} \int_0^{2\pi} K(\eta, \theta)\, \zeta(\eta)\, d\eta$$

which defines purely capillary waves, and which for fixed $2\nu^{(0)} = \nu_n'$ has the unique nontrivial solution $\zeta(\theta) = (C_1 / \sqrt{\pi}) \cos n\theta$ (n is a positive integrer).

Theorem 2.4. For $\varkappa_0 = 2\nu_n''$ it is necessary to set $1/\nu^{(0)} = 0$ (hence, $\mu = 0$). Then instead of Eq. (2.3) we have

$$\tau + A_0 = \nu_n'' \int_0^{2\pi} K_{01}(\eta, \theta)(\tau + A_0)\, d\eta$$

where

$$K_{01}(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{\nu_n''}$$

or

$$\Phi(\theta) = \nu_n'' \int_0^{2\pi} K_0(\eta, \theta)\, \Phi(\eta)\, d\eta$$

for purely gravitational waves which have, respectively, the following unique nontrivial solutions:

$$\tau(\theta) = \frac{C_1}{\sqrt{\pi}} \cos n\theta, \qquad \Phi(\theta) = \frac{C_1}{\sqrt{\pi}} \sin n\theta$$

for fixed integer n. (Allowance is made here $A_0 = 0$ in a linear approximation).

2.2. Solution of the linear problem for $\nu^{(0)} \neq \nu_n$. For analyzing in a linear approximation the possible form of the free surface in terms of wave propagation velocity, we assume the bed line to be specified by

$$\Theta(s) = \varepsilon \sum_{n=1}^{\infty} \beta_n \sin \frac{n\pi s}{l} \tag{2.6}$$

Then

$$\zeta^*(\theta, \varepsilon) = \varepsilon \zeta_1^*(\theta) = -\varepsilon \sum_{i=1}^{\infty} \beta_i \cos i\theta \tag{2.7}$$

Function $\zeta(\theta, \varepsilon)$ is determined by the solution of the related nonhomogeneous linear integral equation derived for $\nu^{(0)} \neq \nu_n$ and is of the form

$$\zeta(\theta, \varepsilon) = \varepsilon 4\nu^{(0)} \sum_{i=1}^{\infty} \frac{\beta_i \nu_i}{\nu_i^* (\nu_i - \nu^{(0)})} \cos i\theta \tag{2.8}$$

Integrating (2.7) and (2.8), we obtain

$$\Phi^*(\theta, \varepsilon) = -\varepsilon \sum_{i=1}^{\infty} \frac{\beta_i}{i} \sin i\theta \tag{2.9}$$

$$\Phi(\theta, \varepsilon) = \varepsilon 4\nu^{(0)} \sum_{i=1}^{\infty} \frac{\beta_i \nu_i}{i\nu_i^* (\nu_i - \nu^{(0)})} \sin i\theta \tag{2.10}$$

Denoting by c_i the wave velocity defined by formula (2.6) for $i = n$, we can show that

$$\nu_i - \nu^{(0)} > 0 \quad \text{for} \quad c < c_i \qquad \nu_i - \nu^{(0)} < 0 \quad \text{for} \quad c > c_i \tag{2.11}$$

Since ν_i and ν_i^* are positive, the signs of coefficients of terms in (2.10) are determined by the sign of $\beta_i \,/\, (\nu_i - \nu^{(0)})$. It follows from (2.11) that the coefficients of related terms in (2.9) and (2.10) have different signs for $c < c_i$, while for $c > c_i$ they are of the same sign. Analyzing expressions (2.9) and (2.10) with allowance for inequality (2.11), we obtain the following results.

If velocity c satisfies the inequalit, $c_{n-1} < c < c_{2n}$, then the free surface crests and troughs lie, respectively, over the crests and troughs of the bed line, while for $c_{2n} < c < c_{2n+1}$, the wave crests and troughs of the free surface lie, respectively, over the troughs and crests of the bed line. It is assumed here that the first term of formula (2.9) determines the shape of the bed line.

For the analysis of solution of the linear problem for $\nu^{(0)} \neq \nu_n$ and the bed line defined by expansion (1.3) see Note 4.3 in Sect. 4 below (see also p. 380 in [1]).

3. Solution of basic equations of the problem. As noted at the end of Sect. 1, the case of $\nu^{(0)} \neq \nu_n$ and that of $\nu^{(0)} = \nu_n$ must be considered separately when solving Eqs. (1.25), (1.27), (1.31), (1.32), and (1.36). We shall indicate the method of solution derivation in both of these. For the first case we shall adduce the results of determination of the first three approximations, and for the second case we shall use, as an example, that of $\nu^{(0)} = \nu_1$ selecting parameter $\varkappa_0$ so as to obtain a simple positive eigenvalue of ν_1 Only the first two approximations have been completely determined, while the determination of the third approximation has not been carried out to the end. For $\nu^{(0)} = \nu_n = \nu_m$ $(n \neq m)$ only the method of solution derivation will be indicated.

3.1. The case of $\nu^{(0)} \neq \nu_n$. As previously noted, the solution is derived in this case in the form of expansions in terms of integral powers of parameter ε. For each coefficient in the expansion of function $\zeta(\theta, \varepsilon)$ we obtain a Fredholm's nonhomogeneous linear integral equation of the second kind with kernel $K^*(\eta, \theta)$ and parameter $\nu^{(0)}$. All these equations are successively solved in accordance with the first Fredholm's theorem. For the determination of coefficients of remaining unknown quantities we obtain a system of linear algebraic equations. From this system, which is always soluble, we derive explicit expressions for the coefficients of a particular approximation in terms of quantities determined in preceding approximations.

The following are the expressions defining $\zeta^*(\theta, \varepsilon)$, $\zeta(\theta, \varepsilon)$, $\delta'(\varepsilon)$ and $A_0(\varepsilon)$ derived in the first three approximations:

$$\zeta^*(\theta, \varepsilon) = -\varepsilon\beta_1 \cos\theta - \varepsilon^2 D_{22} \cos 2\theta - \varepsilon^3 (D_{13} \cos\theta + D_{33} \cos 3\theta)$$

$$\zeta(\theta, \varepsilon) = \varepsilon C_{11} \cos\theta + \varepsilon^2 C_{22} \cos 2\theta + \varepsilon^3 (C_{13} \cos\theta + C_{33} \cos 3\theta) \tag{3.1}$$

$$\delta'(\varepsilon) = -\varepsilon\varkappa_0 C_{11} - \varepsilon^2 \left(\frac{1}{4}\varkappa_0 C_{22} + 2A_{02} + \frac{1}{4}\varkappa_0 C_{11} E_{11}\right) + \varepsilon^3 \delta_3'$$

$$A_0(\varepsilon) = \varepsilon^2 A_{02} = -\frac{1}{4}\varepsilon^2 \left[\left(\frac{\beta_1}{\nu_1'} + \frac{2}{\nu_1^*} C_{11}\right)^2 - \beta_1^2\right]$$

where

$$C_{11} = \frac{4\beta_1 \nu^{(0)} \nu_1}{\nu_1^* (\nu_1 - \nu^{(0)})}, \quad C_{22} = \frac{\nu^{(0)} \nu_2}{\nu_2 - \nu^{(0)}} \left(\frac{4}{\nu_2^*} D_{22} + \frac{3}{4}\varkappa_0 C_{11} E_{11}\right)$$

$$D_{22} = 2\beta_1 \left(\frac{\beta_1}{\nu_1'} + \frac{2}{\nu_1^*} C_{11}\right) + 2\beta_2, \quad E_{11} = -\left(\frac{1}{\nu_1'} C_{11} + \frac{2\beta_1}{\nu_1^*}\right) \tag{3.2}$$

$$E_{22} = -\left(\frac{1}{\nu_2'}C_{22} + \frac{2}{\nu_2^*}D_{22}\right), \quad C_{13} = \frac{\nu^{(0)}\nu_1}{\nu_1 - \nu^{(0)}}C_{13}^*, \quad C_{33} = \frac{\nu^{(0)}\nu_3}{\nu_3 - \nu^{(0)}}C_{33}^*$$

In these expressions C_{13}^* is a linear function of C_{11}^3, $C_{11}^2\beta_1$, $C_{11}\beta_1^2$, $C_{11}C_{22}$, $C_{22}\beta_1$, β_1^3, $C_{11}\beta_2$ and $\beta_1\beta_2$; D_{13} is a linear function of the same arguments as C_{13}^*, except C_{11}^3; C_{33}^* is a linear function of the same arguments as C_{13}^* with the addition of β_3; D_{33} is a linear function of the same arguments as C_{33}^*, except C_{11}^3; δ_3' is a linear function of C_{13}, C_{33}, C_{11}^3, $C_{11}^2\beta_1$, $\beta_1^2 C_{11}$, $C_{11}E_{11}^2$, $C_{11}E_{22}$ and $E_{11}C_{22}$.

3.2. The case of $\nu^{(0)} = \nu_1$. In deriving in this case the solution in the form of expansion in terms of powers of $\varepsilon^{1/3}$, for the first coefficient of expansion of $\zeta(\theta, \varepsilon)$ we obtain a Fredholm's homogeneous linear equation of the second kind for $\nu^{(0)} = \nu_1$. It is solved with the use of Fredholm's second theorem. Equations of subsequent coefficients, although of the smae form, are nonhomogeneous for the same parameter $\nu^{(0)} = \nu_1$. These equations are solved with the use of Fredholm's third theorem. The coefficients of the nth approximate solution of the homogeneous equation is determined by the solvability condition of the equation for the $(n+2)$-nd approximation.

Each of the coefficients C_{11}, C_{12} and C_{13} is successively determined by the related condition of solvability of equations of the third, fourth and fifth approximations. The coefficient C_{13} was not calculated, since we had not determined the fifth approximation. Coefficients of expansions of remaining unknown quantities are determined in the same manner as for $\nu^{(0)} \neq \nu_n$.

The expressions for $\zeta(\theta, \varepsilon)$, $\zeta^*(\theta, \varepsilon)$, $\delta'(\varepsilon)$ and $A_0(\varepsilon)$ derived in the first three approximations are

$$\zeta^*(\theta, \varepsilon) = -\varepsilon\beta_1 \cos\theta$$

$$\zeta(\theta, \varepsilon) = \varepsilon^{1/3} C_{11}\cos\theta + \varepsilon^{2/3} C_{22}\cos 2\theta + \varepsilon(C_{13}\cos\theta + C_{33}\cos 3\theta) \tag{3.3}$$

$$\delta'(\varepsilon) = -\varepsilon^{1/3}\varkappa_0 C_{11} + \varepsilon^{2/3}\left[C_{11}^2\left(\frac{2}{\nu_1^{*2}} + \frac{\varkappa_0}{\nu_1'}\right) - \frac{1}{4}\varkappa_0 C_{22}\right] + \varepsilon\delta_3'$$

$$A_0(\varepsilon) = -\frac{1}{4}\varepsilon^{2/3}\left(\frac{1}{\nu_1'^2} - 1\right)C_{11}^2$$

where

$$C_{11} = -\beta_1^{1/3}\alpha_1^{1/3}, \quad \alpha_1 = \frac{64\nu_1'^2(\nu_2 - \nu_1)(1 - \nu_1'^2)^{1/2}}{(\nu_2 - \nu_1)[8(3 - 2\nu_1'^2) + 12\varkappa_0\nu_1'(1 - \nu_1'^2)] + 9\varkappa_0^2\nu_1\nu_2\nu_1'}$$

$$C_{22} = -\frac{3}{4}\frac{\nu_1\nu_2\varkappa_0}{\nu_1'(\nu_2 - \nu_1)}C_{11}^2, \quad C_{33} = \frac{\nu_1\nu_3}{\nu_3 - \nu_1}C_{33}^* \tag{3.4}$$

In these expressions C_{33}^* is a linear function of C_{11}^3 and $C_{11}C_{22}$; δ_3' is a linear function of C_{13}, C_{33}, C_{11}^3 and $C_{11}C_{22}$; the coefficient $C_{12} = 0$. We recall that in both these cases $\tau(\theta, \varepsilon)$ and $\tau^*(\theta, \varepsilon)$ are determined by (1.14) and (1.37), and $\Phi(\theta, \varepsilon)$ and $\Phi^*(\theta, \varepsilon)$ by (1.38).

3.3. The case of $\nu^{(0)} = \nu_n = \nu_m$ $(n \neq m)$. In this case the solution is derived in the same manner as in the case of $\nu^{(0)} = \nu_n$. The only difference being in that the solution of the homogeneous integral equation in each of the ith approximations contains the sum $C_{in}\cos n\theta + C_{im}\cos m\theta$. In the general case the coefficients C_{in} and C_{im} are determined by the condition of solvability of the equation for the $(i+2)$-nd approximation.

4. Determination of the wave profile. The parametric form of the wave profile $x(\theta, \varepsilon)$ and $y(\theta, \varepsilon)$ is obtained from the relationships (1.12). Passing to dimensionless coordinates x / λ and y / λ and using the same notation, after the substitution of derived $\Phi(\theta, \varepsilon)$ and $\tau(\theta, \varepsilon)$, we obtain the parametric equations of the wave profile. Eliminating in these θ, we obtain the equation of wave profile of the form $y = y(x, \varepsilon)$.

We present below the equations of wave profile, correst to within third order terms, for the two cases, setting $2\pi = k$. For $\nu^{(0)} \neq \nu_n$ we have

$$y(x, \varepsilon) = \frac{1}{k}\Big\{\varepsilon C_{11}(\cos kx - 1) + \frac{1}{4}\varepsilon^2 (C_{22} - E_{11}C_{11})(\cos 2kx - 1) + \frac{1}{6}\varepsilon^3\Big[6C_{13} + \frac{3}{8\nu_1'^2}(3\nu_1'^2 - 4)C_{11}^3 - 6\frac{\beta_1}{\nu_1'\nu_1*}C_{11}\Big(C_{11} - \frac{\beta_1}{\nu_1'\nu_1*}\Big) + \frac{3}{8}C_{11}E_{11}^2 - \frac{3}{2}C_{11}E_{22} + 3C_{22}E_{11}\Big](\cos kx - 1) + \frac{1}{6}\varepsilon^3\Big[\frac{2}{3}C_{33} - \frac{7}{24}C_{11}^3 + \frac{5}{8}C_{11}E_{11}^2 - \frac{1}{2}C_{11}E_{22} - C_{22}E_{11}\Big](\cos 3kx - 1)\Big\} \tag{4.1}$$

where the coefficients C_{ij} and E_{ij} are defined by formulas (3.2).

For $\nu^{(0)} = \nu_1$ we have

$$y(x, \varepsilon) = \frac{1}{k}\Big\{\varepsilon^{1/3}C_{11}(\cos kx - 1) + \frac{1}{4}\varepsilon^{2/3}\Big(C_{22} - \frac{1}{\nu_1'}C_{11}^2\Big)(\cos 2kx - 1) + \frac{1}{6}\varepsilon\Big[6C_{13} + \frac{9}{8}\Big(1 - \frac{1}{\nu_1'^2}\Big)C_{11}^3 + \frac{3}{2}\Big(\frac{2}{\nu_1'} - \frac{1}{\nu_2'}\Big)C_{11}C_{22}\Big](\cos kx - 1) + \frac{1}{6}\varepsilon\Big[\frac{2}{3}C_{33} + \frac{1}{8}\Big(\frac{5}{\nu_1'^2} - \frac{7}{3}\Big)C_{11}^3 - \Big(\frac{1}{\nu_1'} + \frac{1}{2\nu_1'}\Big)C_{11}C_{22}\Big](\cos 3kx - 1)\Big\} \tag{4.2}$$

where the coefficients C_{ij} are defined by formulas (3.4).

Note 1. Since $\zeta_1*(\theta) = -\beta_1 \cos\theta$, the coefficient of the principal term in the expansion of $\Phi*(\theta, \varepsilon)$ is of the form

$$\Phi_1*(\theta) = -\beta_1 \sin\theta \tag{4.3}$$

and, since the coordinate origin is located at the crest of the bed line wave, the conformal image of the coordinate origin is a point at $r = r_0$ and $\theta = 0$. This implies that $\Phi_1*(\theta) > 0$ must correspond to positive θ. Hence we must assume that $\beta_1 < 0$. The angle $\Phi_1(\theta) = C_{11}\sin\theta$ is, also, positive, owing to the assumption that $\nu_1 < \nu^{(0)} < \nu_2$ and because of the expression for C_{11} in the first of formulas (3.2).

Note 4.2. If $\nu^{(0)} = \nu_n$ is the eigenvalue of the kernel of the integral equation, then this is the particular case mentioned at the beginning of this paper. In fact, for $\nu^{(0)} = \nu_n$ from formulas (1.22) and (1.26) we obtain the expression (see formula (2.5)) which in the linear approximation determines the relationship between c and λ_0 in that particular case.

Note 4.3. When $\nu^{(0)} \neq \nu_1$ and the bed line is defined by the expansion (1.3), the analysis of the solution of the linear problem is carried out similarly to that described in 2.2 of Sect. 2, and corresponds to $n = 1$. Subsequent harmonics are taken into account by the addition to the first harmonic in the first term of expansion (1.3) of the sum of n harmonics of order i ($i = 2, 3, \ldots, n$).

The results of the analysis presented in sub-section 2.2 of Sect. 2 can be applied to the investigation of solution of the nonlinear problem, if one considers that the solution of the linear problem determines the principal terms of the complete solution.

5. The existence and uniqueness of solution of the problem. Using the Liapunov-Schmidt methods and their developments [6], we establish the following theorems.

Teorem 5.1. The system of Eqs. (1.25), (1.27), (1.31), (1.32) and (1.36) has for $\nu^{(0)} \neq \nu_n$ a unique solution $\zeta^*(\theta, \varepsilon)$, $\zeta(\theta, \varepsilon)$, $s(\theta, \varepsilon)$, $A_0(\varepsilon)$ and $\delta'(\varepsilon)$ ($\delta'(\varepsilon) = \delta(\varepsilon) - 1$) which is small with respect to ε, continuous with respect to θ ($0 \leqslant \theta \leqslant 2\pi$), and is an analytic function of ε for $|\varepsilon| < \varepsilon_0$.

Theorem 5.2. The system of Eqs. (1.25), (1.27), (1.31), (1, 32) and (1.36) has for $\nu^{(0)} = \nu_1$, where ν_1 is the simple and positive eigenvalue, a unique solution $\zeta^*(\theta, \varepsilon)$, $\zeta(\theta, \varepsilon)$, $s(\theta, \varepsilon)$ $A_0(\varepsilon)$ and $\delta'(\varepsilon)$ which is small with respect to ε and continuous with respect to θ ($0 \leqslant \theta \leqslant 2\pi$), and can be expressed in the form of series expansions in terms of powers of $\varepsilon^{1/3}$ which is convergent for small $|\varepsilon| < \varepsilon_0$.

The proof of these theorems is similar to that carried out in [7, 8]. These theorems imply the absolute and uniform convergence of expansions for $\Phi(\theta, \varepsilon)$, $\tau(\theta, \varepsilon)$, $\Phi^*(\theta, \varepsilon)$ and $\tau^*(\theta, \varepsilon)$. The convergence of expansions in powers of ε and of $\varepsilon^{1/3}$ (for $\nu^{(0)} = \nu_1$) for the integrands of functions in (1.12) follows from the general theorems of the analysis of substitution of series into series. The convergence of expansions whose approximate sums are defined by formulas (4.1) and (4.2) is established on the basis of general theorems of analysis.

BIBLIOGRAPHY

1. Sekerzh-Zen kovich, Ia.I., On the theory of stable capillary-gravitational waves of finite amplitude on the surface of a fluid over an undulating bed; collection: The Application of the Theory of Functions in Mechanics of Continuous Media (in Russian), (Tr. Mezhdunar. Simpoz., Tiblisi, 1963), Vol. 2, "Nauka", Moscow, 1965.
2. Hilbig, H., Existenzbeweis für Potentialströmungen längs eines Kanals mit welliger Kanalsohle unter Einfluss der Schwerkraft und der Oberflächenspannung, Arch. Rat. Mech. Anal., Vol. 18, №5, 1965.
3. Zeildler, E., Beiträge zur Theorie und Praxis freier Randwertaufgaben. Funktionalanalytische Untersuchungen über eine Klasse nichtlinearer hydrodynamischer Probleme. Schriftenr. Inst. Math. Dtsch. Akad. Wiss., Vol. 12, №9, Berlin, 1971.
4. Nekrasov, A.I., The Exact Theory of Waves of Stabilized Form at the Surface of Heavy Fluid (in Russian). Izd. Akad. Nauk SSSR, 1951.
5. Sekerzh-Zen'kovich, Ia.I., On forced stationary capillary-gravitational waves of finite amplitude on the surface of fluid of finite depth. Collection: Mechanics of Continuous Medium and Related Problems of Analysis (in Russian). "Nauka", Moscow, 1972.
6. Vainberg, M.M. and Trenogin, V.A., The methods of Liapunov and Schmidt in the theory of nonlinear equations and their further development. Uspekhi Matem. Nauk, Vol. 17, №2 (104), 1962.
7. Sekerzh-Zen'kovich, Ia.I. On a form of steady waves of finite amplitude. PMM Vol. 32, №6, 1968.
8. Sekerzh-Zen'kovich, Ia.I., On a particular form of steady waves. In book: Theory of Branching of Solutions of Nonlinear Equations. "Nauka", Moscow, 1969.

Translated by J.J.D.